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The non-linear interaction of a finite number of disturbances to a layer of fluid heated from below

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It is predicted that, at a temperature difference a little less than that at which motion starts according to linear stability theory, a steady hexagonal convective pattern will develop from finite-amplitude instabilities in a horizontal layer of fluid heated from below. This is because the first disturbances to start growing must be the triplet of two-dimensional 'rolls' which form angles of 60° with each other and whose amplitudes and phases first fall in certain critical ranges. The growth of these disturbances stabilizes all other disturbances and is such that ultimately the right phases and amplitudes occur to give hexagonal cells. If the temperature difference is increased somewhat beyond its critical value, the hexagonal pattern becomes unstable and a two-dimensional roll pattern is predicted. In an intermediate temperature range, rolls are unstable but transport more heat than hexagons. 'Free–free' boundary conditions, a viscosity which varies with temperature, and a fixed disturbance wave-number are assumed in this extension of the work of Palm (1960) and Segel & Stuart (1962). Other theoretical results and some experimental results are compared with the present predictions.

1. Introduction

We consider here certain aspects of the non-linear stability analysis of a linear temperature profile in a motionless layer of fluid the bottom of which is kept a constant temperature ΔT hotter than the top. Our primary aim is to clarify the mechanism leading to the hexagonal convection cells observed in controlled experiments.

It will be assumed that the reader is familiar with the classical linear stability theory concerned with this problem—as in the book by Chandrasekhar (1961, ch. 2)—and also with the broad outlines of the non-linear investigations of Palm (1960), Segel & Stuart (1962), and Palm & Øiann (1964). We point out, however, that all one needs to know in advance is outlined in the paper preceding this one (Segel 1965) so that the two papers taken together are virtually self-contained. The papers of Palm (1960), Segel & Stuart (1962), and Segel (1965) will henceforth be referred to as I, II, and III.

With a suitable choice of axes, the vertical velocity w for the solution corresponding to hexagonal cells can be taken proportional to

$$2\cos \tfrac{\sqrt{3}}{2}\pi\alpha x \cos \tfrac{1}{2}\pi\alpha y + \cos \pi\alpha y, \tag{1.1}$$

where x and y are horizontal co-ordinates and α is a constant called the overall wave-number. In I and II it was shown that under certain conditions the interaction of two disturbances proportional to

$$\cos \tfrac{\sqrt{3}}{2}\pi\alpha x \cos \tfrac{1}{2}\pi\alpha y \quad \text{and} \quad \cos \pi\alpha y \tag{1.2}$$

leads to a stable finite-amplitude equilibrium state in which the amplitude of the first disturbance is, as in (1.1), precisely twice that of the second. This was encouraging but, leaving aside certain other difficulties to be discussed later, it was not clear why one should limit consideration to the pair of disturbances (1.2). Clarification of this point was the original goal of the work reported here.

We shall show that cellular convection probably commences from a slightly subcritical situation in which the interaction of any finite number of disturbances having the same overall wave-number must result in a stable hexagonal cell. According to our theory, if the Rayleigh number is raised further, somewhat beyond its critical value, the hexagonal equilibrium state becomes unstable and is replaced by a state consisting of two-dimensional 'roll' cells. Comparison with experiment is discussed.

2. Restriction to a single overall wave-number

Because the same growth rate is predicted for all solutions having the same overall wave-number, linear stability theory cannot predict the shape of the convection cells which finally appear after motion commences. In this paper, we shall make a non-linear stability analysis of the motionless state under the assumption that only one overall wave-number and its harmonics need be considered.

This assumption is somewhat supported by experiment although it is not possible to say definitely whether a single wave-number or a narrow band of wave-numbers is observed. While complete understanding of the problem requires an explanation of the overall wave-number selection as well as the cell-shape selection for a given overall wave-number, progress will be made if only the latter is explained. Regarding the former, according to linear stability theory an overall wave-number α can be selected which maximizes the growth rate in a supercritical situation (Rayleigh number $\mathscr{R}$ greater than its minimum critical value $\mathscr{R}_c$) or minimizes the decay rate in a subcritical situation ($\mathscr{R} < \mathscr{R}_c$). Particularly in the latter situation, however, it is not clear that the linear effect dominates. The only relevant theoretical work appears to be that of Segel (1962), where an indication was given of how non-linear terms might act to select a single overall wave-number. For $\mathscr{R}$ slightly greater than $\mathscr{R}_c$ it was shown that two two-dimensional *roll* disturbances, both of which grow exponentially by linear theory, interact in such a manner that one grows to a non-zero limiting amplitude while the other ultimately decays. The interaction of two rolls cannot give rise to a subcritical instability so an extension of this work (in progress) must be made before any conclusions can be drawn when $\mathscr{R} < \mathscr{R}_c$.

Deferring further comment until later, for the reasons just discussed *we assume that in a non-linear stability analysis of the basic temperature profile only first-order disturbances of a certain overall wave-number α need be considered.* We limit this hypothesis to first-order disturbances because when first-order disturbances of overall wave-number α begin to grow, other wave-numbers appear at higher order and must be taken into account. It is to be expected that $|\alpha-\alpha_c|$ is small, but in our analysis the actual value of α is a matter of indifference.

3. Reduction to a six-disturbance analysis

Suppose that the Rayleigh number is slowly raised from well below $\mathscr{R}_c$. There are always small finite perturbations being introduced by irregularities in the experimental apparatus. The most accurate mathematical description of these perturbations may well involve Fourier integrals extending throughout the unbounded (x, y) horizontal plane, but we assume that it is possible to represent the (x, y) dependence by a finite sum of doubly periodic functions. (By properly taking more and more terms of such a sum one can approximate the Fourier integral arbitrarily closely—as is seen at once by considering the definition of the integral as the limit of a sum.) The first-order vertical velocity perturbation w can then be written

$$w = \sum_p [A_p(t)\cos m_p x \cos n_p y + B_p(t)\cos m_p x \sin n_p y + C_p(t)\sin m_p x \cos n_p y + D_p(t)\sin m_p x \sin n_p y] f_p(z). \quad (3.1)$$

When, as in I and II, we assume 'free–free' boundary conditions, $f_p(z)$ has the form

$$f_p(z) = \sin \pi z + K(\alpha_p)\gamma \sin 2\pi z + O(\gamma^2),$$

where K is a constant depending on $\alpha_p \equiv \pi^{-1}(m_p^2+n_p^2)^{\frac{1}{2}}$ and γ is a dimensionless measure of the variation of viscosity with temperature, $|\gamma| \ll 1$. (To facilitate calculations, the law by which the viscosity ν varies with temperature T—defined to be zero at the bottom of the layer—about a reference value ν_0 is assumed, as in I and II, to be

$$\nu/\nu_0 = 1 + \gamma \cos(\pi T/\Delta T) + O(\gamma^2)$$

but qualitative results remain the same for any small variations.) It is frequently convenient to use a formalism slightly different from (3.1), namely

$$W = \sum_p [C_p(t)\cos(m_p x + n_p y) + S_p(t)\sin(m_p x + n_p y) + c_p(t)\cos(m_p x - n_p y) + s_p(t)\sin(m_p x - n_p y)] f_p(z), \quad (3.2)$$

which separates terms associated with the wave-number vector (m_p, n_p) from those associated with the wave-number vector $(m_p, -n_p)$.

Most of the terms in (3.1) and (3.2) have an overall wave-number far different from α_c and so certainly will decay whenever they are temporarily excited. At any rate, by hypothesis only those first-order disturbances with a single overall wave-number α (at or near α_c) need be retained so in (3.1) and (3.2) we need only consider the terms for which

$$m_p^2 + n_p^2 = \pi^2\alpha^2. \quad (3.3)$$

A non-linear stability analysis of these disturbances will lead to *amplitude equations* for the unknown functions of time appearing in (3.1) or (3.2). The equation for a typical amplitude function A_1 will have the form

$$A_1' = \epsilon A_1 + a_{123}\gamma A_2 A_3 + \text{other 2nd-order terms} + \text{3rd-order terms} + \text{negligible terms}. \quad (3.4)$$

The first-order coefficient ϵ will be the same in every case; it is the linear amplification rate appropriate to the overall wave-number α and is proportional to $\mathscr{R}-\mathscr{R}_c$. As illustrated in our analysis of the model equation in III, the second-order terms must be proportional, through an order-one constant like a_{123}, to the small viscosity variation coefficient γ. It is further shown in III that the appearance of the typical second-order term shown explicitly in (3.4) requires that

$$(A_2 \cos m_2 x \cos n_2 y)\,(A_3 \cos m_3 x \cos n_3 y)$$

contain a term proportional to $\cos m_1 x \cos n_1 y$; this *replication requirement* imposes certain conditions on the m's and n's. To find these conditions most easily it will prove helpful to associate an angle with every pair of x and y wave-numbers m_p and n_p. From (3.3) there is a real angle ψ_p such that

$$m_p = \pi\alpha \sin\psi_p, \quad n_p = \pi\alpha\cos\psi_p. \tag{3.5}$$

(In essence, we are using polar co-ordinates to represent the wave-number vector.) We can now state an important result: *If there is a term proportional to A_2A_3 in the equation for A_1' there will be a term proportional to A_1A_2 in the equation for A_3' and a term proportional to A_1A_3 in the equation for A_2'. The three associated angles, ψ_1, ψ_2, and ψ_3, can always be ordered so that the first and second angles are respectively* 60° *and* 120° *greater, than the third angle.* The first part of the result is an obvious consequence of the replication requirement; the second part is proved in appendix 1. Sacrificing some precision to permit a more forceful statement of the result, we assert that second-order terms are associated with triplets of wave-number vectors lying 60° apart.

This is important because, when experimental irregularities are sufficiently small, second-order terms are responsible for any subcritical instabilities which may occür. To see this let δ denote the dimensionless order of magnitude of the inevitable random finite-amplitude perturbations and suppose that $\mathscr{R}$ is gradually raised from well below $\mathscr{R}_c$, which means that ϵ is gradually raised from well below zero. Referring to (3.4), no initial perturbation will grow unless it is such that destabilizing higher-order terms override the stabilizing first-order terms in the amplitude equations. The magnitude of the first-order terms is surpassed by that of the second-order terms when $\mathscr{R}$ is such that $-\epsilon \approx |\gamma|\,\delta$ and by that of the third-order terms when $-\epsilon \approx \delta^2$. If we assume that inevitable experimental irregularities have been so reduced that the dimensionless disturbance magnitude δ is considerably less than the magnitude of the dimensionless viscosity variation coefficient $|\gamma|$, then $|\gamma|\,\delta \gg \delta^2$ and, *as the Rayleigh number is slowly increased, the first instabilities arise when [if it is possible] destabilizing second-order terms outweigh the stabilizing first-order terms.*

We shall see that the second-order terms can have the right signs, as well as sufficient magnitude, to destabilize a situation which is stable according to linearized theory. Consequently, in the course of the random appearance of various disturbances, as $\mathscr{R}$ is slowly raised a certain set of disturbances—associated with a '60° triplet' of wave-number vector angles—will be the first whose second-order terms attain the proper size and sign to give a destabilizing influence exceeding the first-order stabilizing influence. The amplitudes of this set of disturbances will then begin to increase. It might be thought that within a short time after this happens other disturbances associated with other angles will also begin to grow, but it is shown in appendix 2 that third-order terms are stabilizing. When the set of disturbances which first becomes unstable starts to grow, the most dangerous of the other disturbances are just barely stable, their stabilizing first-order terms just barely outweighing their destabilizing second-order terms. The third-order terms, although negligible compared to the first- and second-order terms individually, are then not negligible compared to the algebraic sum of these terms. The growth of the small third-order terms then has the significant qualitative effect of never permitting the growth of even the most dangerous disturbances not in the initially growing set. The stabilizing third-order terms will also ultimately limit the amplitude of disturbances in the initially growing set. We therefore expect a final equilibrium state composed only of those modes, associated with a single triplet of angles, which are the first to start growing.

Although a person interested only in the principal features of our argument can pass at once to § 4, it might be helpful to examine the result just obtained from a slightly different point of view. With each angle $\psi_j = \sin^{-1}(m_j/\pi\alpha)$ we have associated two modes, proportional to

$$C_j(t)\cos(m_j x + n_j y) \quad \text{and} \quad S_j(t)\sin(m_j x + n_j y),$$

so each triplet of angles is associated with six modes. Equivalently, we could have associated each angle with one mode,

$$R_j(t)\cos[m_j x + n_j y + \theta_j(t)],$$

a roll of amplitude $R_j(t)$ and phase $\theta_j(t)$. From this point of view our assertion is that the final subcritical convective state is determined by the interaction of those three rolls, making angles of 60° with each other, whose amplitudes and phases first fall in certain critical ranges.

It may seem unlikely that random disturbances in a very large layer will be in the form of a roll. Doubt concerning this point can probably be removed by recalling our assumption that the functions we encounter can be expanded in series like (3.2), in which case any disturbance can be synthesized, to a sufficiently good approximation, from a large number of rolls.

One's intuitive picture of how cellular orientation is determined might be that orientations are randomly selected in various patches with some sort of accommodation taking place as convective patterns from various patches spread and merge. For this picture to be valid, other triplets of rolls in addition to the first would have to start growing before the stabilizing influence associated with the growth of the first triplet has time to become significant. We assume that the experimental irregularities are such that this does not happen, which requires that the disturbance growth time be short compared to a representative correlation time of the perturbations. We are trying to explain the nearly uniform cell patterns sometimes observed. That such observations are rare is not surprising in view of the stringent conditions which appear to be required for the emergence of a single pattern over the entire layer.

To obtain a mathematical view of the mechanism under discussion in this section, consider a phase space in which the amplitude of each mode is represented on a co-ordinate axis. This phase space can be regarded as the direct sum of six-dimensional subspaces on whose co-ordinate axes are represented the six modes connected with a triplet of angles. Before any disturbances start growing, each individual amplitude is very small and varies independently of the others. The point representing the state of the system (its co-ordinates at any time giving the amplitude of the various modes) moves randomly near the origin. Growth starts when the projection of this state-point in one of the six-dimensional subspaces first enters a region of the subspace, somewhat away from the origin, where destabilizing second-order terms outweigh stabilizing first-order terms. Then the projection of the state-point in this subspace moves away from the origin. Due to the growth of stabilizing terms in the amplitude equations the projection of the state-point in all other subspaces moves toward the origin. Because of the continual exterior disturbances to the fluid layer, the state-point still undergoes random forced vibrations but the tendency of the motion just sketched will not be altered if these vibrations are sufficiently small.

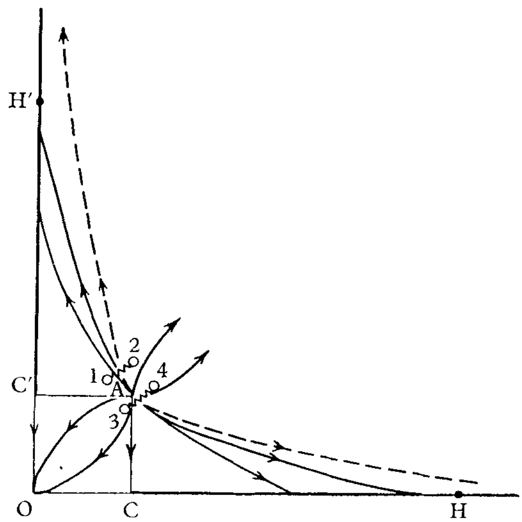

FIGURE 1. Model of twelve-dimensional phase space when $\mathscr{R} < \mathscr{R}_c$. Axes represent six-dimensional subspaces. H and H′ represent stable hexagonal equilibrium states toward which almost all trajectories starting a little outside C′AC tend. Jumps from 1 to 2 or 3 to 4 represent possible ways to avoid H and H′.

The behaviour we have deduced of possible trajectories in phase space can be visualized in a simple case if we imagine the positive quadrant of the (x, y)-plane to be a model for an entire twelve-dimensional phase space, and the two positive co-ordinate axes to represent two six-dimensional subspaces. In figure 1 we have represented with a heavier line those portions of the co-ordinate axes which correspond to regions where destabilizing second-order terms can outweigh stabilizing first-order terms. The above discussion shows that the trajectories starting near the origin must behave as sketched. Random effects force the state-point across AC or AC′, starting it on its journey to one of the six-dimensional subspaces represented by OCH or OC′H′. A comparatively unlikely [by our assumption on disturbance growth and correlation times] forced jump across the dotted trajectories, as from 1 to 2 or from 3 to 4, is the only way to avoid a final state in OCH or OC′H′. By determining the nature of the second- and third-order terms in the amplitude equations we have thus been able to establish, conclusively but without detailed analysis, certain facts about the qualitative behaviour of trajectories in phase space. The state-point representing disturbances which are the first to start growing from a slightly subcritical situation is almost certain to approach an equilibrium point in one of the six-dimensional subspaces. The problem is thus reduced to a study of trajectories in one of these subspaces.

4. Equations for the six disturbance amplitudes

As is shown at the end of appendix 1, by a rotation of the co-ordinate axes we can arrange it so that analysis of the trajectories in the six-dimensional subspace associated with the first set of disturbances to start growing is equivalent to following the development of a perturbation which, to lowest order, has the form

$$W = \phi(x,y,t)\,g(z),$$

$$\begin{aligned}\phi \equiv\; & U(t)\sin\tfrac{\sqrt{3}}{2}\pi\alpha x\sin\tfrac{1}{2}\pi\alpha y + V(t)\cos\tfrac{\sqrt{3}}{2}\pi\alpha x\sin\tfrac{1}{2}\pi\alpha y + W(t)\sin\pi\alpha y\\ & + X(t)\sin\tfrac{\sqrt{3}}{2}\pi\alpha x\cos\tfrac{1}{2}\pi\alpha y + Y(t)\cos\tfrac{\sqrt{3}}{2}\pi\alpha x\cos\tfrac{1}{2}\pi\alpha y + Z(t)\cos\pi\alpha y. \qquad (4.1)\end{aligned}$$

Very fortunately, it is possible to derive the six required amplitude equations indirectly by following the procedure of III. We outline the steps briefly. By computing ϕ^2 and multiplying all terms of the same overall wave-number by the same constant, we find that the second-order terms obtained on substituting the first-order solution $\phi(x,y,t)\,g(z)$ into the non-linear portion of the governing partial differential equations are, for some constants C_i,

$$\begin{aligned}\Phi =\; & C_0[\tfrac{1}{4}(U^2+V^2+X^2+Y^2)+\tfrac{1}{2}(W^2+Z^2)]\,F_0(z)\\ & + C_1[-(UZ-WX)\sin\tfrac{\sqrt{3}}{2}\pi\alpha x\sin\tfrac{1}{2}\pi\alpha y-(VZ-WY)\cos\tfrac{\sqrt{3}}{2}\pi\alpha x\sin\tfrac{1}{2}\pi\alpha y\\ & + \tfrac{1}{2}(YV+XU)\sin\pi\alpha y+(XZ+UW)\sin\tfrac{\sqrt{3}}{2}\pi\alpha x\cos\tfrac{1}{2}\pi\alpha y\\ & + (YZ+VW)\cos\tfrac{\sqrt{3}}{2}\pi\alpha x\cos\tfrac{1}{2}\pi\alpha y+\tfrac{1}{4}(X^2+Y^2-U^2-V^2)\cos\pi\alpha y]\,F_1(z)\\ & + C_3[\tfrac{1}{2}UV\sin\sqrt{3}\,\pi\alpha x+\dots]\,F_3(z)+C_4[\tfrac{1}{2}UY\sin\sqrt{3}\,\pi\alpha x\sin\pi\alpha y+\dots]\,F_4(z),\end{aligned}$$

where we have written out only one of the many terms of overall wave-numbers $\sqrt{3}\,\alpha$ and 2α. The form of the F's need not be specified. The C_1 terms replicate U, V, W, X, Y, Z, respectively, and so, for some constant a proportional to γ, give rise to the second-order terms in (4.2) below. To find the third-order terms in the amplitude equations, we determine the replicating terms in $\Phi\phi$, for example

$$(\tfrac{1}{16})\,U^2X\sin\tfrac{\sqrt{3}}{2}\pi\alpha x\cos\tfrac{1}{2}\pi\alpha y[4C_0+2C_1+2C_3+C_4].$$

We finally obtain the six amplitude equations

$$U' = \epsilon U + a[UZ - WX] - U[RX^2 + R_2Y^2 + PZ^2 + RU^2 + RV^2 + PW^2] - \tfrac{1}{2}QVXY, \quad (4.2)$$

$$V' = \epsilon V + a[VZ - WY] - V[R_2X^2 + RY^2 + PZ^2 + RU^2 + RV^2 + PW^2] - \tfrac{1}{2}QUXY, \quad (4.3)$$

$$W' = \epsilon W - \tfrac{1}{2}a[YV + XU] - W[\tfrac{1}{2}PX^2 + \tfrac{1}{2}PY^2 + R_1Z^2 + \tfrac{1}{2}PU^2 + \tfrac{1}{2}PV^2 + R_1W^2], \quad (4.4)$$

$$X' = \epsilon X - a[XZ + UW] - X[RX^2 + RY^2 + PZ^2 + RU^2 + R_2V^2 + PW^2] - \tfrac{1}{2}QUVY, \quad (4.5)$$

$$Y' = \epsilon Y - a[YZ + VW] - Y[RX^2 + RY^2 + PZ^2 + R_2U^2 + RV^2 + PW^2] - \tfrac{1}{2}QUVX, \quad (4.6)$$

$$Z' = \epsilon Z - \tfrac{1}{4}a[X^2 + Y^2 - U^2 - V^2] - Z[\tfrac{1}{2}PX^2 + \tfrac{1}{2}PY^2 + R_1Z^2 + \tfrac{1}{2}PU^2 + \tfrac{1}{2}PV^2 + R_1W^2]. \quad (4.7)$$

For some constants C_0, C_1, C_3 and C_4:

$$R = \tfrac{1}{16}(4C_0 + 2C_1 + 2C_3 + C_4), \quad R_1 = \tfrac{1}{4}(2C_0 + C_4),$$

$$R_2 = \tfrac{1}{16}(4C_0 - 2C_1 - 2C_3 + 3C_4), \quad Q = \tfrac{1}{4}(2C_1 + 2C_3 - C_4), \quad P = \tfrac{1}{2}(C_0 + C_1 + C_3),$$

so that
$$R_2 = R_1 - R, \quad Q = 4R - 2R_1, \quad P = 4R - R_1. \quad (4.8)$$

With comparatively little work we have obtained six amplitude equations, through third order, which contain three unknown constants a, R, and R_1. Various results can be found without knowing anything about these three constants, but a complete discussion of (4.2)–(4.7) requires their determination. This is done most easily by setting $U = V = W = X = 0$ and performing a standard *two*-disturbance analysis to obtain the remaining equations

$$Y' = \epsilon Y - aYZ - Y(RY^2 + PZ^2), \quad Z' = \epsilon Z - \tfrac{1}{4}aY^2 - Z(\tfrac{1}{2}PY^2 + R_1Z^2), \quad (4.9\,a, b)$$

in which a, R, and R_1 appear. But this two-disturbance analysis has already been done in I and II. Equations (4.9a) and (4.9b) are precisely equations (3.3) and (3.4) of II; a, R, and R_1 are given in equations (3.6), (2.19) and (2.20) of II. [The factor $(1+\alpha_1^2)$ in the first term on the right-hand side of II, equation (2.19), should be $(1+\alpha_1^2)^2$.]

If we evaluate ϵ, a, R, and R_1 at $\alpha^2 = \frac{1}{2}$, $\mathscr{R} = 27\pi^4/4$ (the linear theory critical values in the absence of viscosity variation with temperature), we can obtain an idea of their size. We find that

$$\epsilon = 0{\cdot}0225\mathscr{P}_1(\mathscr{R} - \mathscr{R}_c), \quad a = 0{\cdot}567\mathscr{P}_1\gamma,$$

$$100R = [0{\cdot}665\mathscr{P}^{-2} + 0{\cdot}428\mathscr{P}^{-1} + 8{\cdot}02]\mathscr{P}_1, \quad R_1 = 0{\cdot}125\mathscr{P}_1, \quad (4.10)$$

where $\mathscr{P}$ is the Prandtl number, $\mathscr{P}_1 \equiv \mathscr{P}(1+\mathscr{P})^{-1}$, and γ is the dimensionless measure of viscosity variation with temperature defined in § 3. We can thus see that P, Q, R, and R_1 are positive in the region of interest but that R_2 is negative if the Prandtl number has a value below about $\frac{1}{3}$. (The time and velocity-amplitude scales used in making (4.2)–(4.7) and (4.9) dimensionless are h^2/κ and κ/h, where κ is the thermal conductivity and h is the thickness of the layer.)

Although the six amplitude equations (4.2)–(4.7) are not tidy-looking, they actually possess considerable symmetry. The symmetry is present because, although possible rotations of the co-ordinate system are now ruled out, a reference angle having been determined by the wave-number vectors of the first subcritical disturbances to begin growing, translations of the axes are still possible but should not alter our conclusions. It makes the algebra a little simpler if we consider the effects of a translation through $2\xi/\sqrt{3}$ in the x-direction and 2η in the y-direction:

$$\tfrac{\sqrt{3}}{2}x = \tfrac{\sqrt{3}}{2}x_1+\xi, \quad \tfrac{1}{2}y = \tfrac{1}{2}y_1+\eta. \tag{4.11}$$

In the translated variables, with subscript unity, the basic disturbance becomes

$$\begin{aligned}\phi_1(x_1, y_1, t) = {} & U_1 \sin \tfrac{\sqrt{3}}{2}\pi\alpha x_1 \sin \tfrac{1}{2}\pi\alpha y_1 + V_1 \cos \tfrac{\sqrt{3}}{2}\pi\alpha x_1 \sin \tfrac{1}{2}\pi\alpha y_1 \\ & + W_1 \sin \pi\alpha y_1 + X_1 \sin \tfrac{\sqrt{3}}{2}\pi\alpha x_1 \cos \tfrac{1}{2}\pi\alpha y_1 \\ & + Y_1 \cos \tfrac{\sqrt{3}}{2}\pi\alpha x_1 \cos \tfrac{1}{2}\pi\alpha y_1 + Z_1 \cos \pi\alpha y_1,\end{aligned}$$

which has the same form as $\phi(x, y, t)$. If C denotes a column vector with elements U, V, W, X, Y, and Z, the connexion between C and C_1 is given by

$$C_1 = M(\xi, \eta)\, C, \tag{4.12}$$

where $M(\xi, \eta)$ is the matrix

$$\left|\begin{array}{cccccc} \cos\xi\cos\eta & -\sin\xi\cos\eta & -\cos\xi\sin\eta & \sin\xi\sin\eta & 0 & 0 \\ \sin\xi\cos\eta & \cos\xi\cos\eta & -\sin\xi\sin\eta & -\cos\xi\sin\eta & 0 & 0 \\ \cos\xi\sin\eta & -\sin\xi\sin\eta & \cos\xi\cos\eta & -\sin\xi\cos\eta & 0 & 0 \\ \sin\xi\sin\eta & \cos\xi\sin\eta & \sin\xi\cos\eta & \cos\xi\cos\eta & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos 2\eta & -\sin 2\eta \\ 0 & 0 & 0 & 0 & \sin 2\eta & \cos 2\eta \end{array}\right|. \tag{4.13}$$

Since the translation of axes can be performed in either order by separate x and y translations, we are not surprised that the matrix associated with this translation is orthogonal and has the properties

$$\left.\begin{aligned} M(\xi, \eta) &= M(\xi, 0)\, M(0, \eta) = M(0, \eta)\, M(\xi, 0), \\ \det M(\xi, \eta) &= [\det M(\xi, 0)]\,[\det M(0, \eta)] = 1 \times 1 = 1. \end{aligned}\right\} \tag{4.14}$$

A little further investigation shows three quantities to be invariant under the translation

$$U^2+V^2+X^2+Y^2 = U_1^2+V_1^2+X_1^2+Y_1^2, \quad W^2+Z^2 = W_1^2+Z_1^2,$$
$$UY-VX = U_1Y_1-V_1X_1.$$

The equations satisfied by the components of C_1 can be obtained by combining (4.2)–(4.7) according to the transformation (4.12). The new equations are identical with the old; this is to be anticipated, but provides a good check on the correctness of the original amplitude equations (4.2)–(4.7).

An alternative version of (4.2)–(4.7) more clearly shows the symmetries present. Instead of (4.1) we can write the first set of disturbances to start growing as

$$\Phi = A_1(t)\cos[\pi\alpha y+\theta_1(t)]+A_2(t)\cos[\pi\alpha(\tfrac{1}{2}y-\tfrac{\sqrt{3}}{2}x)-\theta_2(t)] \\ +A_3(t)\cos[\pi\alpha(\tfrac{1}{2}y+\tfrac{\sqrt{3}}{2}x)-\theta_3(t)], \quad (4.15)$$

where the six unknown functions are now the amplitudes $A_i(t)$ of rolls, inclined at angles of 60° with one another, having phase $\theta_i(t)$; $i = 1, 2, 3$. Comparing (4.1) and (4.15) we find

$$A_1^2 = W^2+Z^2, \quad A_2^2+A_3^2 = \tfrac{1}{2}(U^2+V^2+X^2+Y^2), \quad A_2^2-A_3^2 = YU-XV, \quad (4.16)$$

so (4.14) has the interpretation that the amplitudes of the rolls are invariant under axis-translation. We also find

$$\theta_1 = -\tan^{-1}(W/Z), \quad \theta_2 = \tan^{-1}[(V-X)/(U+Y)], \\ \theta_3 = \tan^{-1}[(V+X)/(Y-U)]. \quad (4.17)$$

Using (4.16) and (4.17), equations (4.2)–(4.7) can be manipulated to give

$$\left.\begin{aligned} A_1' &= \epsilon A_1 - aA_2A_3\cos(\theta_1+\theta_2+\theta_3) - A_1[R_1A_1^2+PA_2^2+PA_3^2],\\ A_2' &= \epsilon A_2 - aA_1A_3\cos(\theta_1+\theta_2+\theta_3) - A_2[PA_1^2+R_1A_2^2+PA_3^2],\\ A_3' &= \epsilon A_3 - aA_1A_2\cos(\theta_1+\theta_2+\theta_3) - A_3[PA_1^2+PA_2^2+R_1A_3^2],\\ A_1\theta_1' &= aA_2A_3\sin(\theta_1+\theta_2+\theta_3),\\ A_2\theta_2' &= aA_1A_3\sin(\theta_1+\theta_2+\theta_3),\\ A_3\theta_3' &= aA_1A_2\sin(\theta_1+\theta_2+\theta_3). \end{aligned}\right\} \quad (4.18a\text{–}f)$$

The symmetry evidenced in (4.18) is to be expected because of the lack of distinction among the three interacting rolls. It is further explained by the fact that the arguments of the cosines in (4.15) vanish along three lines which cut out an equilateral triangle whose sides have length $2\sqrt{3}(\theta_1+\theta_2+\theta_3)$ [a translation-invariant quantity], so that permutation of indices on the amplitude and phase functions does not change the geometrical picture represented by (4.15). From (4.18) we learn that the phases $\theta_i(t)$ only change in our model because of the viscosity variation with temperature ($a \neq 0$). In contrast to (4.2)–(4.7), (4.18) brings out the completely stabilizing nature of third-order terms obtained on interacting the three rolls of (4.15).

5. The equilibrium solutions and their stability

The next step is to find the possible equilibrium solutions of the system (4.2)–(4.7), or of (4.18) (see appendix 3). In a subcritical situation the only possible non-trivial solutions turn out to be a two-parameter family of hexagonal cells, the existence of a two-parameter family of equilibrium solutions being due to the presence of the two-parameter family of possible axis-translations. In a supercritical situation, the possible equilibrium solutions are as follows:

(*a*) the hexagonal cells just discussed, (*b*) roll cells aligned along the x-axis or at angles of $\pm 60°$ to the x-axis,

(*c*) a specie of generally closed cell like that labelled V on p. 294 of II.

A local stability analysis reveals that the cells of (*c*) are unstable. (Some details of these and subsequently mentioned analyses are given in appendix 4.) When $|\mathscr{R}-\mathscr{R}_c|$ is sufficiently small, a local stability analysis of the hexagonal cells appears at first sight to show that they are neutrally stable. Four of the six characteristic roots of the resulting determinant are negative, corresponding to a four-dimensional manifold of locally decaying solutions. Two characteristic roots are zero. These would normally be ascribed to solutions which appear locally neutral but whose actual local growth or decay can only be determined by considering terms of higher order than those retained in a local linearization.

$\epsilon < -\epsilon_{-1}$	Undisturbed state
$-\epsilon_{-1} < \epsilon < 0$	Undisturbed state and hexagonal cells
$0 < \epsilon < \epsilon_1$	Hexagonal cells
$\epsilon_1 < \epsilon < \epsilon_2$	Hexagonal cells and rolls
$\epsilon_2 < \epsilon$	Rolls

$$\epsilon_{-1} \equiv a^2/4T, \quad \epsilon_1 \equiv a^2Q^{-2}R_1, \quad \epsilon_2 = a^2Q^{-2}(4R+R_1).$$

TABLE 1. Stable situations for various ranges of ϵ (which is proportional to $\mathscr{R}-\mathscr{R}_c$). The Rayleigh number $\mathscr{R}$ increases as one reads down the table. The solid line divides subcritical ($\mathscr{R}<\mathscr{R}_c$) from supercritical ranges. The quantity a is a dimensionless measure of viscosity variation with temperature, assumed small. ϵ, a, R, R_1, Q and T are defined in (4.10), (4.8) and (A 3.18).

The situation considered here is unusual, however, in that the equilibrium points are not isolated but occur on a two-dimensional *equilibrium surface* in the six-dimensional phase space. Consider any given equilibrium point. A trajectory which starts from a nearby point on the equilibrium surface will neither approach the given equilibrium point nor recede from it; a trajectory through a point on the equilibrium surface consists entirely of that point. This is the reason for the two zero characteristic roots. Since the other four characteristic roots are negative, we would expect that the equilibrium *surface* is stable in that any trajectory which starts near the equilibrium surface ultimately reaches it. This expectation has been shown correct by N. Levinson (private communication).

A slightly different way of viewing the matter starts with the observation that a local stability analysis of a given equilibrium state examines the growth or decay of a state which is initially 'near' the given equilibrium state. Take any particular hexagonal pattern as the given state. A nearby state which initially consists wholly or in part of a slightly translated pattern will approach the translated pattern, not the given pattern. (An imprecise view that our results show the hexagonal pattern to be neutral with respect to instantaneous translations might lead to the incorrect deduction that the zero characteristic roots will not appear when one or both stress-free boundary conditions are replaced by boundary conditions appropriate to solid walls.)

Once the significance of the zero characteristic roots is appreciated, stability determination becomes a routine matter. The results are given in table 1. We see that, if the Rayleigh number is slowly and continuously raised, hexagonal cells appear but then disappear, giving way to rolls aligned in one of the three directions determined by the hexagons' boundaries.

6. Possible effects due to modes other than the six considered

We have shown that in a subcritical situation a convective motion must be made up of disturbance modes (and their harmonics) associated with a triplet of wave-number vectors lying 60° apart. Which particular triplet appears depends on which disturbances are by chance the first whose second-order mutual destabilization outweighs first-order stabilizing effects. The final convective state must be a hexagonal pattern whose angular orientation is the same as that of the triplet selected by chance. We have further shown that this hexagonal pattern is stable to all six disturbances associated with the selected triplet, but not that it is stable to other disturbances.

Let us therefore consider a disturbance, proportional to $A(t)$ say, whose associated wave-number vector is not in the selected triplet. What we have learned concerning the situation when second-order terms appear, and (in appendix 2) when indefinite third-order terms appear, shows that there are no terms like AU or AVW or XYZ in the A' amplitude equation. Linearizing this equation about an equilibrium point $(u_0, v_0, w_0, x_0, y_0, z_0)$ therefore gives

$$A' = [\epsilon - S_1 u_0^2 - S_2 v_0^2 - S_3 w_0^2 - S_4 x_0^2 - S_5 y_0^2 - S_6 z_0^2] A, \tag{6.1}$$

where the S_i are positive constants given by Palm & Øiann (1964). In a subcritical state the square bracket in (6.1) is negative so the hexagonal pattern is in fact stable to all disturbances.

What happens to the stability of the hexagonal pattern when the Rayleigh number is slowly raised to a slightly supercritical value? It is easy to see that the pattern remains stable as long as ϵ is small compared to a^2. This is because (see II, figure 1) when $\epsilon \ll a^2$ equilibrium amplitudes of stable hexagons are proportional to a. Consequently when ϵ is positive but small compared to a^2, its destabilizing effect in (6.1) is outweighed by the stabilizing effects of the other terms. By a numerical examination of the terms in (6.1) for the special but representative case $u_0 = v_0 = w_0 = x_0 = 0$, Palm & Øiann (1964) conclude in § 5 of their paper that even when ϵ is not small no destabilizing influence arises from (6.1). If we consider the stability of the roll

$$u_0 = v_0 = w_0 = x_0 = y_0 = 0, \quad z_0^2 = \epsilon/R_1,$$

(6.1) becomes
$$A' = \epsilon R_1^{-1}[R_1 - S_6] A.$$

From equation (2.8) of Palm & Øiann (1964), where S_6 is denoted by T, it follows that $R_1 - S_6$ is negative so again no destabilizing influence arises from (6.1). We conclude that table 1 summarizes correctly the stability behaviour of hexagons and rolls subject to all disturbances of the wave-number to which we are restricting ourselves. It is interesting that at no value of $\mathscr{R}$ can an instability develop due to a mode other than the six directly associated with the given hexagon or roll.

When ϵ is positive we can no longer assert, as in § 3, that a trajectory starting near the origin remains in the six-dimensional subspace where it starts. Although we do not expect it, we cannot rule out the possibility of an equilibrium state composed of modes associated with many different wave-number vectors. On the other hand, a hexagonal pattern will inevitably appear in a subcritical state as a result of finite amplitude instabilities. Until ϵ is raised to the supercritical value ϵ_2 this pattern is stable to sufficiently small disturbances. Finite amplitude disturbances may grow—indeed we know that they can lead to rolls if $\epsilon > \epsilon_1$—but, subject as always to possible consequences of relaxing our restriction to one overall wave-number, we can assert that in sufficiently careful experiments hexagonal cells will appear when $|\mathscr{R}-\mathscr{R}_c|$ is small.

We conjecture that the ultimate reason why our restriction to one overall wave-number is justified might be that, as is suggested to some extent by the results of Segel (1962), the initial growth of a set of disturbances of one overall wave-number puts stabilizing terms into the amplitude equations for disturbances of all other wave-numbers. A completely general analysis would then be determined by an analysis confined to a single overall wave-number in the same way that a single overall wave-number analysis is determined by the six disturbance analysis of §§ 4 and 5.

7. Heat transfer

The graph of the predicted heat transfer is given in figure 2. The Nusselt number N (the ratio of the actual horizontally averaged heat transfer, which is

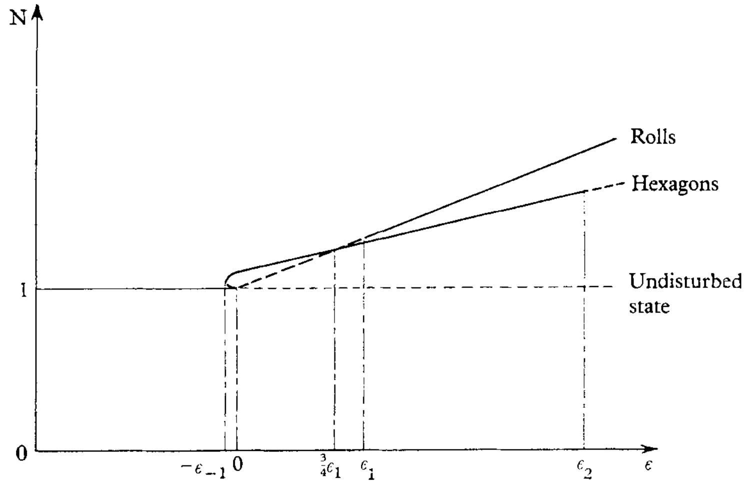

FIGURE 2. Dimensionless heat transfer N plotted against ϵ, a quantity proportional to $\mathscr{R}-\mathscr{R}_c$. Note that before they are a stable solution rolls transport more heat than hexagons. ——, Stable; – – –, unstable.

independent of height, to the heat transfer which would occur by conduction alone) is plotted against ϵ. N can be computed from

$$N = \mathscr{R}^{-1}(dT/dz)_{z=0},$$

where T is the horizontally averaged temperature. The only stable equilibrium states can be represented by $U = V = W = X = 0$ so we can use a dimensionless version of the formula for D_{002} in I (6.14), as corrected in II, p. 293, to write

$$-\mathscr{R}^{-1}\, d\mathrm{T}/dz = 1+\tfrac{1}{4}\pi^{-2}(\alpha^2+1)^{-1}(\tfrac{1}{2}Y^2+Z^2)\cos 2\pi z+\ldots.$$

For hexagons, $Y = \pm 2Z$ and $Z = \zeta$, a constant defined in (A 3.6). For rolls, $Y = 0$ and $Z^2 = \epsilon/R_1$.

With reference to figure 2, the following points should be noted.

(i) In the ranges of Rayleigh number where two stable modes are possible, a hysteresis effect should be observed in sufficiently accurate heat-transfer experiments. For example, as soon as ϵ exceeds ϵ_1 hexagons can become replaced by rolls but relatively large finite amplitude disturbances are required. If the temperature difference between the bounding planes is slowly increased, the actual transition would be expected to take place for ϵ somewhat closer to ϵ_2, above which infinitesimal disturbances cause hexagons to be replaced by rolls. Similarly, if the temperature difference is slowly decreased, rolls should be replaced by hexagons when ϵ is near ϵ_1. The smaller one can keep the inevitable finite-amplitude disturbances to the fluid, the more pronounced will be the observed hysteresis. In a private communication, J. T. Stuart previously pointed out this phenomenon by observing the position in the (Y, Z)-plane of (A 3.4), the equilibrium point called V in II. For $\epsilon_1 < \epsilon < \epsilon_2$, V is a saddle-point which lies on the separatrix dividing trajectories approaching the roll equilibrium point from those approaching the hexagonal equilibrium point. As ϵ increases from ϵ_1 up to ϵ_2 this separatrix can be seen to move from arbitrarily near the roll equilibrium point to arbitrarily near the hexagonal equilibrium point, so that an ever smaller disturbance to the hexagons will cause the separatrix to be crossed and rolls to appear.

(ii) The interesting effects depicted (instability at Rayleigh number below critical, heat-transfer jumps due to finite-amplitude instability, hysteresis) are prominent in proportion to the markedness of fluid property variation with temperature. Prandtl number effects enter through the constants ϵ_i. As $\mathscr{P}\to 0$ all the various Rayleigh numbers distinguished in figure 2 approach $\mathscr{R}_c$. As $\mathscr{P}\to\infty$, these Rayleigh numbers approach asymptotic values independent of $\mathscr{P}$ (see (4.10) and § 9).

(iii) For a range of Rayleigh numbers at which rolls transfer more heat than hexagons, rolls are an unstable equilibrium mode but hexagons are stable. There is no contradiction here with the relative stability theory of Malkus & Veronis. Their result that the stable finite-amplitude solution, of the type we consider, will maximize heat transfer was derived for the Boussinesq equations where $\gamma = 0$. As a matter of fact, Lortz has shown (private communication) that for the Boussinesq equations the heat transfer does have a maximum for rolls so there is agreement with Malkus & Veronis (1958) if, contrary to the preference of these authors, one does not discard the rolls as 'unphysical' compared to limiting rectangles. (Both could be correct idealizations in appropriate instances, depending on whether one or many long thin 'rectangular' cells appear to span the width of a large convecting layer.) We note that, although the above remarks lend some support to the relative stability criterion, Herring (1964) has pointed out that, even when the initial decay of a disturbance is predicted by Malkus & Veronis's relative stability arguments, later growth may take place. Further, figure 2 shows that the relative stability concept is not correct for $\mathscr{R}$ near $\mathscr{R}_c$ when equations more precise than those of Boussinesq are considered. It appears that relative stability should be considered a suggestive concept whose validity must be separately investigated in each new situation.

The fact that there is a range of $\mathscr{R}$ for which unstable rolls transport more heat than stable hexagons can also be compared with Malkus's maximum heat transfer hypothesis (Howard 1963), but it should be borne in mind that this hypothesis was formulated primarily to deal with $\mathscr{R}$ large while our result is for $\mathscr{R}$ in a range near $\mathscr{R}_c$. Figure 2 is similar to figure 7 of Veronis's (1959) study of finite-amplitude cellular convection in which an imposed uniform rotation plays a similar role to the variation of viscosity with temperature in the present work. Veronis predicted subcritical effects corresponding to those found here, but was not able to give firm theoretical backing to his predictions since his steady-state calculations were not carried far enough and since no stability determination was made of the equilibrium points found. For certain ranges of the parameters involved, Veronis's predictions have been verified in some unpublished work of J. Watson.

Veronis (1963) has discussed another situation in which subcritical instabilities are expected, namely, a layer of water whose top and bottom temperatures are respectively colder and hotter than 4 °C, at which temperature the density of water is a maximum. Once again, subcritical effects corresponding to those found here were predicted but only partially verified.

In Veronis's (1959) rotating fluid study, he states that 'under experimental conditions the two fluids, mercury and air, which are considered...will not exhibit...finite amplitude instability'. In water near 4 °C there is some evidence, not conclusive, that subcritical instabilities have occurred in experiments by Furumoto & Rooth (1961). The possibility of observing subcritical instabilities caused by property variations with temperature is discussed in § 9.

8. Comments on some recent papers

In this section we sketch the relationships among this and other recent papers dealing with cell shape in thermal convection.

When he had made considerable progress in the present work but before the computations needed for the final results were completed, the author saw an early version of Palm & Øiann (1964). It was most helpful, in § 6 and in appendix 2, to be able to use the values they had calculated for certain constants in the amplitude equations. It must also be mentioned that certain of the results derived (independently) here were anticipated by Palm & Øiann (1964). For example, by a brief elegant argument they show that the stability of rolls, found in II for $0 < \epsilon < \epsilon_1$ when Y and Z modes are considered, is destroyed when a V mode is introduced. [A V mode is the disturbance in (4.1) with amplitude $V(t)$.] To give another example, Palm & Øiann (1964) consider the stability of hexagonal cells (where $Y = \pm 2Z$) to V and W modes, but do not completely appreciate the significance of the zero characteristic root which they find. In neither of the examples cited do they investigate the new equilibrium solutions which appear when new modes are considered.

Bisshopp (1960) has shown that there is a one-parameter family of solutions to the *linear* thermal stability problem, proportional to

$$2\cos\tfrac{1}{2}k\sqrt{3}\,x\cos\tfrac{1}{2}ky+\cos(ky+\theta), \tag{8.1}$$

which have hexagonal symmetry. To various values of the parameter θ there correspond genuinely different flow patterns. Equation (8.1) fits into our six disturbance analysis if we take, for some constant ξ,

$$k=\pi\alpha,\quad Y=2\xi,\quad Z=\xi\cos\theta,\quad W=-\xi\sin\theta,\quad U=V=X=0,$$

but if $U=V=0$ then (4.3) shows that unless $a=0$ we must have $WY=0$. Consequently, when we take variation of viscosity with temperature into account, there is a solution of the *non-linear* equations proportional at lowest order to (8.1) only when $\theta=0$. When $\theta=0$, (8.1) represents the hexagonal cell already considered.

Considerable work on thermal convection appears in Inaugural Dissertations, done under the guidance of A. Schlüter, by Lortz (1961) and Busse (1962). This work is important and closely connected with the topic of this paper, and much of it is unpublished, so we shall summarize its principal features before commenting on it. Notation already introduced will be used as much as is practicable.

Busse (1962) studies equations in which the viscosity q_1, thermal conductivity q_2, specific heat at constant pressure q_3, and thermal expansion coefficient q_4 are assumed to vary with temperature T according to laws of the form

$$q_i=q_{i0}[1+\gamma_i(T-T_0)+O(\gamma_i^2)]\quad (i=1,2,3,4), \tag{8.2}$$

where T_0 is the mean of the fixed temperatures T_2 and T_1 of the top and bottom bounding planes located a distance h apart. He shows that terms proportional to the bulk viscosity have no effect and assumes that the quantity $gh/(T_1-T_2)\,q_{30}$ is very small so that the dissipation and changes of the q_i with pressure can be neglected.

The main part of Busse's work is an extension of Lortz's (1961) discussion of the standard Boussinesq equations, for which the q_i are constants. In most of what follows we do not distinguish between the contributions of Lortz and Busse and, for brevity, mention only the farthest-reaching results of the two dissertations.

In the style of the pioneering work of Malkus & Veronis (1958) a *steady* solution—other than the motionless conductive state—is sought by expanding the velocity and temperature fields and the Rayleigh number in double power series in ζ and ϵ, the series for the vertical velocity and the Rayleigh number taking the form

$$w=\sum_{i=1,j=0}^{\infty}\epsilon^i\zeta^j w_{ij},\quad \mathscr{R}=\sum_{i,j=0}^{\infty}\epsilon^i\zeta^j\mathscr{R}_{ij}. \tag{8.3 a, b}$$

(To save writing, we mention only w of the three velocity components and the temperature.) Here ζ denotes the common order of magnitude of the γ_i, and ϵ is an amplitude parameter which can be determined from (8.3 b) since $\mathscr{R}$ and the $\mathscr{R}_{ij}$ will be known.

A second major part of the work is a linear stability analysis of the steady solution obtained. Assuming a time dependence $\exp(\sigma t)$, one obtains a linear homogeneous problem for the eigenvalue σ. The functions multiplying the differential operators in this problem come from the steady solution and hence are double power series in ϵ and ζ. One can therefore write, for σ and the vertical velocity eigenfunction $\tilde{w}$,

$$\tilde{w} = \sum_{i=1,j=0}^{\infty} \epsilon^i \zeta^j \tilde{w}_{ij}, \quad \sigma = \sum_{i,j=0}^{\infty} \epsilon^i \zeta^j \sigma_{ij}. \tag{8.4a, b}$$

Once the steady solution is determined to a certain order, its stability can be analysed to the same order. In practice, terms of a given order in w and $\mathscr{R}$ and in $\tilde{w}$ and σ are computed simultaneously.

At order ϵ, one is in essence confronted with the standard self-adjoint-type linear stability problem of the Boussinesq equations. $\mathscr{R}_{00}$ is taken to be the minimum critical Rayleigh number $\mathscr{R}_c$ and α to be the corresponding overall wave-number α_c. (This is for definiteness: the results to be found are true for any α if $\mathscr{R}_{00}$ is taken to be the corresponding critical Rayleigh number.) The most dangerous disturbance has overall wave-number α_c and a zero growth rate, so $\sigma_{00} = 0$. The eigenfunction indeterminancy of linear theory is dealt with by keeping the analysis very general at this stage. It is assumed that

$$w = \sum_{k=-N}^{N} c_k e^{i(m_k x + n_k y)} f(z) \quad \text{where} \quad m_k^2 + n_k^2 = \pi^2 \alpha_c^2, \quad (m_{-k}, n_{-k}) = -(m_k, n_k), \tag{8.5a, b, c}$$

$$c_k = c^*_{-k}, \quad \sum_{k=-N}^{N} |c_k|^2 = 1, \tag{8.6a, b}$$

but the complex constants c_k are not specified further. The series for $\tilde{w}_{10}$ has the same form as that for w_{10}, with coefficients $\tilde{c}_k$.

The equations for higher-order approximations are inhomogeneous. For solutions to exist, the inhomogeneous terms of these equations must be orthogonal to all solutions of the (self-adjoint) homogeneous equations. At $O(\epsilon^2\zeta^0)$ this *existence condition* gives $R_{10} = \sigma_{10} = 0$. At $O(\epsilon^3\zeta^0)$ the existence condition and the normalization condition (8.6*b*) are shown to lead to an inhomogeneous set of $N+1$ equations which completely determines the constants $|c_k|$ and R_{20} once the wave-number vectors (m_k, n_k) are specified. For *regular solutions*, defined to have the property that the angles between neighbouring wave-number vectors (m_k, n_k) are equal, it is indicated that the $O(\epsilon^i\zeta^j)$ existence conditions not already considered ($i \geqslant 4, j \geqslant 1$) lead to no new restrictions. The same holds for *semi-regular solutions* formed by superposing one regular solution with another rotated through an angle ψ, $\psi \neq \frac{1}{3}\pi$. A detailed examination of the secular determinants giving the values of σ_{20} associated with each of the possible steady solutions based on (8.5)—no restriction to regular or to semi-regular solutions is made—shows that, if no ζ terms are considered, only the rolls are stable for symmetric (free–free or fixed–fixed) boundary conditions. For asymmetric boundary conditions, the $O(\epsilon^4)$ terms are such that (formally) hexagons can also be stable even if $\zeta = 0$, but this first happens when $\mathscr{R}$ is $3{\cdot}4\mathscr{R}_c$, a value at which the convergence of the series is problematical.

At $O(\epsilon\zeta)$ the effect of ζ is not felt since for all boundary conditions the existence conditions require $R_{01} = \sigma_{01} = 0$. For asymmetric boundary conditions, this result is true because reference values for the viscosity, etc., are taken at a certain determined height, not at the middle of the layer as for the other boundary conditions. Therefore, if $\mathscr{R}$ is based on the average viscosity, etc., an $O(\zeta)$ effect on $\mathscr{R}_c$ will generally be observed for asymmetric boundary conditions. At $O(\epsilon^2\zeta)$ the existence condition requires $R_{11} = \sigma_{11} = 0$ except for hexagonal cells and superpositions of hexagonal cells. It is stated that further examination shows that only an equally weighted superposition of two hexagonal cells can in fact exist, but this result is not necessary because all superpositions of hexagons are later shown unstable. For hexagons, R_{11} is given by

$$\zeta R_{11} = A_0 \alpha + \sum_{i=1}^{4} A_i \gamma_i. \tag{8.7}$$

The constants A_i depend only on the Prandtl number. Their numerical value is given for free–free boundary conditions and for the other boundary conditions in the case of infinite Prandtl number. σ_{11} is shown to be such that when $\zeta \neq 0$, for all boundary conditions, as the Rayleigh number is raised the motionless state is succeeded by hexagons and then by rolls in a way described accurately by table 1 except for the actual parameter values at which the transitions take place.

In a general comparison of the Lortz–Busse work and that of the present paper, the following points should be made.

(i) The Lortz–Busse work is much more comprehensive in its consideration of general property variation with temperature and of several different types of boundary condition. On the other hand, the same qualitative results are found in all cases. It is gratifying that whenever they can be compared there is agreement among the results of Palm & Øiann (1964), Lortz (1961), Busse (1962), and the present author.

(ii) The analyses of Lortz–Busse and the present author are largely confined to a finite number of disturbances of the same overall wave-number—compare (8.5*a*) with (3.2) and (3.3)—although Busse does investigate perturbations of a wave-number slightly different from that of the steady solution. He finds bands of stable overall wave-numbers, proportional in length to $|\epsilon|$.

(iii) The connexion between possible translations and zero disturbance growth rates is not mentioned by Busse so his results as written only show the non-instability of rolls and hexagons.

(iv) In the time-varying approach of the present paper, the behaviour of the final non-linear amplitude equations is determined by a local stability analysis of the equilibrium points. This is equivalent to the Lortz–Busse linearized stability analysis of possible steady solutions. On the other hand, the time-varying approach allows the possibility of a more exact analysis of the amplitude behaviour, for example, the determination of about how long it takes for disturbances to develop (as in § 9) or a numerical or Liapounoff analysis of the relative size of the stability regions for rolls and hexagons when they are both stable to infinitesimal disturbances. This approach also seems to give more understanding of why the results come out as they do. Examples of this occur in our discussions of how convection begins from a subcritical state and how the orientation of the convective pattern is determined by the initial growth of one randomly determined set of disturbances and the concurrent stabilization of all other disturbances.

Time-dependent non-linear convection studies, using computers to integrate some truncated set of resulting non-linear ordinary differential equations, are becoming increasingly prevalent in meteorology and oceanography. A detailed analytic study of an idealized problem, such as the one considered here, ought to give insights as to what might happen in more realistic situations. As an example, the understanding we have gained of how the growth of some disturbances can stabilize others might be useful in explaining observations of the enhancement of some cloud streets and the suppression of others (Whitney 1961, p. 459).

9. Comparison with experiment

The most extensive experimental results with which we can compare the theoretical predictions are those of Silveston (1958) on the silicone oil AK 350. This very viscous substance has a Prandtl number of about 3500 so we can simplify our formulas to the infinite Prandtl number case. In fact, as a glance at (4.10) will show, the infinite Prandtl number coefficients are still a good approximation for Prandtl numbers somewhat under 10.

In Silveston's experiments, the silicone oil was held between two plates 7 mm apart and convection set in when the temperature of the bottom plate was about 40 °C and that of the top plate about 20 °C. From table 2 of Silveston's paper, the ratio of the viscosity difference between top and bottom to the mean viscosity was therefore about $-150/400$. From the viscosity variation law in § 3, this ratio can be taken to be 2γ. Therefore γ in this case is about $-\frac{1}{5}$. (We note here that Silveston's thermal expansion coefficients are consistently 10 times what they should be; this slip is reproduced in Chandrasekhar 1961, p. 66.)

For infinite Prandtl number, by computing ϵ_{-1} one sees that finite-amplitude instabilities can occur when $\mathscr{R}$ is about $7\gamma^2$ below $\mathscr{R}_c$ while, from equation (5.9) in I, $\mathscr{R}_c$ itself is lowered by about $130\gamma^2$ from its constant viscosity value. The same result, that the linear effect contributes about 95 % of the total lowering, from its constant property linear theory value, of the Rayleigh number at which convection can start, was made in the discussion of gases in II. With $\gamma^2 = \frac{1}{25}$, motion is predicted to set in at a Rayleigh number about 1 % less than the free–free constant property value of about 660. Since Silveston's careful experiments could only predict $\mathscr{R}_c$ to within 3 %, it is not surprising that not even the lowering of $\mathscr{R}_c$ predicted by linear theory was observed.

When $\mathscr{P} = \infty$, the theory states that rolls can first appear when $\mathscr{R} - \mathscr{R}_c = 350\gamma^2$ and must appear when $\mathscr{R} - \mathscr{R}_c = 1270\gamma^2$ (where the hexagons become unstable). For $\gamma^2 = \frac{1}{25}$ the rolls ought to appear by the time $\mathscr{R}$ is about 50 units, or about 8 %, above critical, while Silveston's figure 12 indicates that the rolls appear at about $\mathscr{R} = 2200$ or about 30 % above the critical value of 1700 for fixed–fixed boundaries. The theoretical result is therefore of the right order of magnitude but quite a bit too low. On the other hand, quantitative agreement is not to be expected between a theory for free–free boundary conditions and experiments for fixed–fixed boundary conditions. Moreover, in Silveston's observations of changing cell patterns, the top copper plate used in his heat transfer measurements was replaced by a glass one, making the actual boundary condition a good deal different from the one of infinite conductivity which we have assumed.

Examination of figure 9 of Silveston's paper leads one to a possible reason why rolls appear at higher $\mathscr{R}$ than that predicted. This figure shows convective patterns at various Rayleigh numbers, and also gives the time since heating began. In 10 min, $\mathscr{R}$ increases from 1700 to 1900. But, if one makes a crude computation of the time it takes disturbances to develop by considering

$$U = V = W = X = Y = 0, \quad dZ/(\epsilon Z - R_1 Z^3) = dt, \tag{9.1}$$

one finds that the rolls of (9.1) develop from $\frac{1}{10}$ to $\frac{9}{10}$ of their equilibrium value in $150h^2/(\mathscr{R}-\mathscr{R}_c)\kappa$ time units. Putting in the proper values of the distance between the planes ($h = 0{\cdot}7$ cm) and thermal conductivity ($\kappa = 4\,\mathrm{cm}^2/\mathrm{h}$) this gives a development time of twelve minutes at $\mathscr{R}-\mathscr{R}_c = 100$. A determination of the hexagon–roll transition from Silveston's figure 9 would therefore overestimate the transition Rayleigh number by quite a bit, since the temperature difference would have increased significantly before the transition would have had time to develop naturally. Another piece of evidence for this point of view is J. T. Stuart's observation (private communication) that if

$$w = Y \cos \tfrac{\sqrt{3}}{2}\pi\alpha x \cos \tfrac{1}{2}\pi\alpha y + Z \cos \pi\alpha y \quad (Z \gg Y),$$

which corresponds to a situation in which the hexagon has not quite developed into a roll, the cells are infinitely long but have slightly wavy boundaries. This may correspond to the 'wormlike' shapes seen in some of Silveston's pictures.

To summarize, there seems to be a measure of agreement for the Rayleigh number at which hexagons change into rolls. There has been no theoretical prediction of this transition until recently so it is not surprising that past experimenters have noted it somewhat qualitatively as they passed on to higher Rayleigh numbers. A quantitative comparison of theory and experiment now seems practicable.

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Appendix 1

Triplets of disturbances reinforcing each other at second order

Remembering that our discussion is confined to modes of overall wave-number α, we examine in detail how second-order terms appear in the amplitude equations. An understanding of the replication principle (see III, §2) is assumed.

It is convenient here to write, as in (3.2), the two disturbances corresponding to a single wave vector (m_1, n_1) as

$$\cos(m_1x+n_1y) \quad \text{and} \quad \sin(m_1x+n_1y). \tag{A 1.1}$$

Using polar co-ordinates and (3.5),

$$x = r\cos\theta, \quad y = r\sin\theta, \quad m_1 = \pi\alpha\sin\psi_1, \quad n_1 = \pi\alpha\cos\psi_1, \tag{A 1.2}$$

we note that
$$m_1x+n_1y = \pi\alpha r\sin(\psi_1+\theta). \tag{A 1.3}$$

Let us consider two disturbances described by two different wave-number vectors. The interaction of disturbances proportional to

$$\cos(m_1x+n_1y) \quad \text{and} \quad \cos(m_2x+n_2y), \tag{A 1.4}$$

say, gives rise to a term proportional to

$$\left.\begin{aligned} &2\cos(m_1x+n_1y)\cos(m_2x+n_2y) \\ &\quad = 2\cos[\pi\alpha r\sin(\psi_1+\theta)]\cos[\pi\alpha r\sin(\psi_2+\theta)] \\ &\quad \equiv \cos\{\pi\alpha r[\sin(\psi_1+\theta)+\sin(\psi_2+\theta)]\}+\cos\{\pi\alpha r[\sin(\psi_1+\theta)-\sin(\psi_2+\theta)]\}.\end{aligned}\right\} \tag{A 1.5}$$

If this interaction of two modes of overall wave-number α is to replicate a third mode of the same overall wave-number, one of the terms on the right-hand side of (A 1.5) must have overall wave-number α. If Φ has overall wave-number α then, in polar co-ordinates,

$$\Phi_{rr}+r^{-1}\Phi_r+r^{-2}\Phi_{\theta\theta}+\pi^2\alpha^2\Phi = 0. \tag{A 1.6}$$

If, from (A 1.5), $\quad \Phi = \cos\{\pi\alpha r[\sin(\psi_1+\theta)\pm\sin(\psi_2+\theta)]\},$

(A 1.6) turns out to require

$$[\sin(\psi_1+\theta)\pm\sin(\psi_2+\theta)]^2+[\cos(\psi_1+\theta)\pm\cos(\psi_2+\theta)]^2 = 1, \tag{A 1.7}$$

or
$$\cos(\psi_1-\psi_2) = \mp\tfrac{1}{2}. \tag{A 1.8}$$

Let us focus our attention on the possibility of a positive sign in (A 1.8). This means that we are considering the second term on the right-hand side of (A 1.5). We associate this term with the subscript 3:

$$\cos\{\pi\alpha r[\sin(\psi_1+\theta)-\sin(\psi_2+\theta)]\}\equiv\cos(m_3x+n_3y)\equiv\cos[\pi\alpha r\sin(\psi_3+\theta)].$$

From (A 1.8) we thus find
$$\psi_2 = \psi_1\pm\pi/3. \tag{A 1.9}$$

Since
$$\sin(\psi_1+\theta)-\sin(\psi_1\pm\pi/3+\theta) = \sin(\psi_1\mp\pi/3+\theta),$$

(A 1.9) implies that

$$\cos(m_3x+n_3y) = \cos\pi\alpha[x\cos(\psi_1\mp\pi/3)+y\sin(\psi_1\mp\pi/3)]. \tag{A 1.10}$$

Consequently one possibility is that disturbances proportional to $\cos(m_1x+n_1y)$ and $\cos(m_2x+n_2y)$ interact to put second-order terms in the equation for the amplitude function associated with $\cos(m_3x+n_3y)$, where ψ_2 is related to ψ_1 by the alternatives given in (A 1.9) and, from (A 1.10),

$$\psi_3 = \psi_1\mp\pi/3. \tag{A 1.11}$$

If we take the negative sign in (A 1.8) we find another possibility

$$\psi_2 = \psi_1\pm 2\pi/3, \quad \psi_3 = \psi_1\pm\pi/3. \tag{A 1.12}$$

If either or both of the terms in (A 1.4) are sines, it is easy to see that all possibilities are still encompassed by (A 1.9), (A 1.11) and (A 1.12). We can thus say that there is a one-parameter family of six interacting disturbances, associated with three wave-number angles, whose first-order horizontal dependence is given by

$$\begin{matrix}\cos\\ \sin\end{matrix}[\pi\alpha r\sin(\psi+\theta)],\quad \begin{matrix}\cos\\ \sin\end{matrix}[\pi\alpha r\sin(\psi-\pi/3+\theta)],\quad \begin{matrix}\cos\\ \sin\end{matrix}[\pi\alpha r\sin(\psi+\pi/3+\theta)]. \tag{A 1.13}$$

In other words, *disturbances interact with each other at second order only if they are associated with the same angle triplet* ψ, $\psi-\pi/3$, $\psi+\pi/3$.

If only one angle ψ is of interest, it can be chosen to be zero by the rotation of axes
$$\theta\rightarrow\theta-\psi,$$

giving six disturbances proportional to

$$\begin{matrix}\cos\\ \sin\end{matrix}[\pi\alpha y],\quad \begin{matrix}\cos\\ \sin\end{matrix}[\pi\alpha(\tfrac{1}{2}y-\tfrac{\surd 3}{2}x)],\quad \begin{matrix}\cos\\ \sin\end{matrix}[\pi\alpha(\tfrac{1}{2}y+\tfrac{\surd 3}{2}x)], \tag{A 1.14}$$

where we have changed back to Cartesian co-ordinates. Reverting to our earlier notation where the horizontal dependence is given in the form $\cos mx \cos ny$, by taking sums and differences of the last four disturbances in (A 1.14) we can write the six disturbances as

$$\begin{matrix}\cos\\ \sin\end{matrix}[\pi\alpha y],\quad \begin{matrix}\cos\\ \sin\end{matrix}[\tfrac{1}{2}\pi\alpha y]\begin{matrix}\cos\\ \sin\end{matrix}[\tfrac{\sqrt{3}}{2}\pi\alpha x], \tag{A 1.15}$$

or, in the form of (4.15), as rolls with varying phases.

Appendix 2

The nature of third-order terms in the amplitude equations

Consider the amplitude equation

$$A_1'(t) = \ldots - C_1 A_1 A_2^2 - C_2 A_1 A_2 A_3 - C_3 A_2 A_3 A_4 - \ldots, \tag{A 2.1}$$

where C_1, C_2, and C_3 denote constants and only certain terms have been written out. The first of these terms always either stabilizes or destabilizes, depending on whether C_1 is positive or negative. The effect of the second and third terms varies with the changing signs of $A_1(t)$, $A_2(t)$, $A_3(t)$, and $A_4(t)$ (see III, § 3). Terms like the first will be called *definite*; terms like the second and third, *indefinite*. In discussing the appearance of definite and indefinite third-order terms, let us denote by $E(mx+ny)$ either $\cos(mx+ny)$ or $\sin(mx+ny)$.

Definite third-order terms occur widely, for, since

$$[A_1(t)\, E(m_1 x \pm n_1 y)]^2 A_i(t)\, E(m_i x \pm n_i y) = \tfrac{1}{2} A_1^2 A_i E(m_i x \pm n_i y) + \text{other terms},$$

there will be a definite third-order term proportional to $A_1^2 A_i$ in every A_i' amplitude equation. The constant of proportionality has been computed by Palm & Øiann (1964) and it is easily shown from their work that these definite third-order terms are stabilizing, at least when ϵ is small and α is not too far from α_c.

We now show that at third order indefinite third-order terms never appear if, as in (3.2), the (x, y) dependence is taken proportional to $E(mx+ny)$. The interaction of disturbances proportional to

$$E(m_1 x \pm n_1 y),\quad E(m_2 x \pm n_2 y)\quad \text{and}\quad E(m_3 x \pm n_3 y)$$

gives rise to terms of the form

$$E[(m_1 \pm m_2 \pm m_3)x \pm (n_1 \pm n_2 \pm n_3)y]. \tag{A 2.2}$$

Consider one such term,

$$E[(m_1 - m_2 + m_3)x + (n_1 + n_2 - n_3)y]. \tag{A 2.3}$$

If this term is to replicate a term of overall wave-number α we must have

$$(m_1 - m_2 + m_3)^2 + (n_1 + n_2 - n_3)^2 = \pi^2\alpha^2,$$

or, using the polar co-ordinate representation of our wave-number vectors of length $\pi\alpha$,

$$-m_1 m_2 + n_1 n_2 + m_1 m_3 - n_1 n_3 - m_2 m_3 - n_2 n_3 = -1,$$

$$\cos(\psi_1 + \psi_2) - \cos(\psi_1 + \psi_3) - \cos(\psi_2 - \psi_3) = -1.$$

If we put $$\xi \equiv \psi_1+\psi_2, \quad \eta \equiv \psi_1+\psi_3,$$
we have $$\cos\xi-\cos\eta-\cos(\xi-\eta) = -1, \tag{A 2.4}$$
or $$(1-\cos^2\eta)^{\frac{1}{2}}(1-\cos^2\xi)^{\frac{1}{2}} = (1-\cos\eta)(1+\cos\xi).$$
Either $$1+\cos\xi = 0, \quad \text{or} \quad 1-\cos\eta = 0, \tag{A 2.5a, b}$$
or $$(1+\cos\eta)(1-\cos\xi) = (1+\cos\xi)(1-\cos\eta),$$
which gives $$\cos\xi = \cos\eta, \quad \text{so, from (A 2.4)}, \quad \xi = \eta. \tag{A 2.6}$$
The three possibilities of (A 2.5a, b) and (A 2.6) give
$$\begin{aligned} &\xi = \pi; && \psi_1 = \pi-\psi_2; && m_1 = m_2, && n_1 = -n_2; \\ &\eta = 0; && \psi_1 = -\psi_3; && m_1 = -m_3, && n_1 = n_3; \\ &\xi = \eta; && \psi_2 = \psi_3; && m_2 = m_3, && n_2 = n_3. \end{aligned}$$
The same sort of results follow from all combinations of signs in (A 2.2). Hence, if three modes interact to replicate a fourth and all modes have the same overall wave-number, then it is necessary that one of the interacting modes have x and y wave-numbers of the same absolute value as those of at least one other.

Let us examine more closely an interaction involving two modes having the same wave-numbers. The interaction of $A_2(t)\cos(m_1x+n_1y)$, $A_3(t)\sin(m_1x+n_1y)$ and $A_4(t)\sin(m_2x+n_2y)$ gives rise, among others, to a term proportional to $\cos(m_3x+n_3y)$, where $m_3 = 2m_1-m_2$ and $n_3 = 2n_1-n_2$. If this term is to have overall wave-number α then
$$(2m_1-m_2)^2+(2n_1-n_2)^2 = \pi^2\alpha^2$$
or, again using the polar co-ordinate representation of our wave-number vectors of length $\pi\alpha$,
$$\cos(\psi_1-\psi_2) = 1; \quad \psi_1 = \psi_2; \quad m_1 = m_2 = m_3, \quad n_1 = n_2 = n_3;$$
and we have a special case of the definite terms already discussed. Once again the situation explicitly considered turns out to be entirely typical so that we can assert that if the (x, y) dependence is taken proportional to $E(mx+ny)$ as in (3.2) then third-order terms are definite and stabilizing. If other forms of (x, y) dependence are taken, the definite character of individual third-order terms need not be preserved. Indeed, although all third-order terms in (4.18) are definite, all those in (4.2)–(4.7) are not. (It can be shown that, with the representation (3.1), indefinite terms appear only if all four of the modes involved have x wave-numbers of the same magnitude and hence y wave-numbers of the same magnitude.) On the other hand, the net third-order stabilizing effect persists no matter what equivalent representation for the disturbances is used.

Appendix 3

Possible equilibrium states

It is much easier to find equilibrium solutions to (4.18) than to the equivalent set (4.2)–(4.7). We obtain first
$$A_2 = A_3 = 0; \theta_1, \theta_2, \theta_3 \text{ arbitrary}; \quad A_1^2 = \epsilon/R_1;$$
which, from (4.15), is a one-parameter family of rolls
$$\Phi = A_1\cos[\pi\alpha y+\theta_1] \tag{A 3.1}$$

aligned parallel to the y-axis. With $\theta_1 = 0$, this is identical with the equilibrium point labelled II in reference II, p. 294. From the symmetry of (4.18) it follows that there are two other one-parameter families of equilibrium solutions which are rolls making angles of $\pm 60°$ with (A 3.1). Superposition of two rolls at equilibrium is not possible, for $A_3 = 0$ and $A_1, A_2 \neq 0$ leads to the contradiction

$$\cos(\theta_1+\theta_2+\theta_3) = \sin(\theta_1+\theta_2+\theta_3) = 0.$$

If $A_1, A_2, A_3 \neq 0$, (4.18d, e, f) imply

$$\sin(\theta_1+\theta_2+\theta_3) = 0 \quad \text{so} \quad \cos(\theta_1+\theta_2+\theta_3) = \pm 1. \tag{A 3.2a, b}$$

As an example of how the calculations proceed, suppose $a > 0$ and take the minus sign in (A 3.2b). Multiplying (4.18a) by A_2 and (4.18b) by $-A_1$, and adding, gives

$$(aA_3 - QA_1A_2)(A_1^2 - A_2^2) = 0.$$

If $A_1^2 \neq A_2^2$, substitution of aA_3Q^{-1} for A_1A_2 in (4.18c) yields

$$\epsilon + a^2Q^{-1} = PA_1^2 + PA_2^2 + R_1A_3^2.$$

If $A_1^2 \neq A_3^2$ and $A_2^2 \neq A_3^2$ two similar equations are obtained and one finds a contradiction to the assumption that A_1^2, A_2^2, A_3^2 are all unequal.

If $A_2^2 = A_3^2$, suppose $A_2 = A_3$. Multiplying (4.18b) by $-A_1A_2^{-1}$ and adding (4.18a) gives

$$(A_1^2 - A_3^2)(a - QA_1) = 0. \tag{A 3.3}$$

If the second factor in (A 3.3) is zero we obtain

$$A_1 = aQ^{-1}, \quad A_2^2 = A_3^2 = (4R)^{-1}(\epsilon + a^2Q^{-1} - Pa^2Q^{-2}) = (4R)^{-1}(\epsilon - R_1a^2Q^{-2}). \tag{A 3.4}$$

Since θ_1, θ_2, and θ_3 are required only to satisfy the single relation (A 3.2a), this is a two-parameter family of equilibrium solutions. When $\theta_1 = \theta_2 = \theta_3 = 0$ we have the closed cell labelled V in reference II, p. 294; other members of the two-parameter family correspond to translations of the same equilibrium pattern. Solutions of this type with $A_2, A_3 = aQ^{-1}$ correspond to the same pattern rotated by $\pm 60°$.

If the first factor of (A 3.3) is zero we have

$$A_1^2 = A_2^2 = A_3^2, \quad A_1 = (2T)^{-1}[a \pm \sqrt{(a^2 + 4\epsilon T)}], \tag{A 3.5}$$

which, together with

$$A_1^2 = A_2^2 = A_3^2, \quad A_1 = (2T)^{-1}[-a \pm \sqrt{(a^2 + 4\epsilon T)}] \equiv \zeta, \tag{A 3.6}$$

obtained by taking the plus sign in (4.20b), represents the hexagonal cells labelled III and IV in reference II, p. 294, and translations thereof.

Appendix 4

Local stability of the equilibrium states

It is best to use (4.2)–(4.7) for stability calculations, because the non-linear terms are polynomials and because earlier results can be utilized. As an example of the notation and procedure to be used in this appendix, we will denote an equilibrium value of $X(t)$ by x_0, we will write $X(t) = x_0 + x(t)$, and we will neglect non-linear terms involving quantities like $x(t)$ in determining the local stability of a given equilibrium point $(u_0, v_0, w_0, x_0, y_0, z_0)$. Rather than consider a general equilibrium point in each class, we will consider a particular equilibrium point for which as many as possible of the quantities $(u_0, v_0, w_0, x_0, y_0, z_0)$ are zero. This simplifies the calculations and is permissible because an axis translation cannot affect stability.

For example, instead of considering the stability of the general hexagonal equilibrium point we will consider the stability of the special case

$$y_0 = \pm 2z_0, \quad z_0 = \pm \zeta,$$

the hexagonal equilibrium point discussed in II. (ζ is the constant given in (A 3.6).) We find, making simplifications by means of the equations satisfied by y_0 and z_0,

$$x' = 0, \quad y' = -2Ry_0^2 y - y_0(a+2Pz_0)z, \quad z' = -\tfrac{1}{2}y_0(a+2Pz_0)y - z_0(2R_1 z_0 + a)z,$$
$$u' = (2az_0 + \tfrac{1}{2}Qy_0^2)u, \quad v' = 2az_0 v - ay_0 w, \quad w' = -\tfrac{1}{2}ay_0 v + az_0 w.$$

The y' and z' equations are the same as those considered in II so we need only concern ourselves with the other four equations, which are independent of y and z. The four new characteristic roots are

$$0, 0, 3az_0, 2z_0(a+Qz_0).$$

As discussed in § 5, the two zero roots are expected; neither of the other two are positive when stability was predicted in II, so there is no change in the results of II concerning stability of hexagons.

The situation is otherwise for the generally closed cells, associated with (A 3.4), which exist when ϵ is positive and sufficiently large. Again extending the analysis of II where $u_0 = v_0 = w_0 = x_0 = 0$, we find that one of the four new roots is

$$-(4R)^{-1}(Q^2\epsilon a^{-2} + T)az_0. \tag{A 4.1}$$

Now in general for this class of equilibrium solutions, solving (4.7) for az_0 shows that there is no ambiguity in the sign of az_0. With $u_0 = v_0 = 0$ this sign is negative so the root given in (A 4.1) is positive and these closed cells are always unstable. Stability was possible according to the two-disturbance analysis of II.

Analysis of the stability of the roll whose only non-zero element is z_0, where $z_0^2 = \epsilon/R_1$, gives the six characteristic roots

$$0, -2\epsilon, \epsilon \pm az_0 - Pz_0^2 \quad \text{(two identical roots for each choice of sign).} \tag{A 4.2}$$

That there is only one zero root is due to the fact that the roll proportional to $z_0 \cos \pi\alpha y$, being unaffected by x-translations, is a member of the one-parameter family of equilibrium solutions corresponding to possible y-translations. From (A 4.2), a necessary and sufficient condition for stability, regardless of the sign in $z_0 = \pm\sqrt{(\epsilon/R_1)}$, is

$$\epsilon > a^2Q^{-2}R_1.$$